星系天文学中的统计基础

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Contents

- Statistical Modelling of data
  - Distribution function, e.g. luminosity function
  - Extreme value statistics
  - Stacking

- Physical modelling of data
  - Correlations ?
  - Linear relations

Statistical view of the world

- All measurements have uncertanity
  - D: data/Measurement
  - M: Model/Fact
  - Bayesian approach: P(D|M) --> P(M|D)
- World/Fact may also be statistical
- Model the data/world statistically

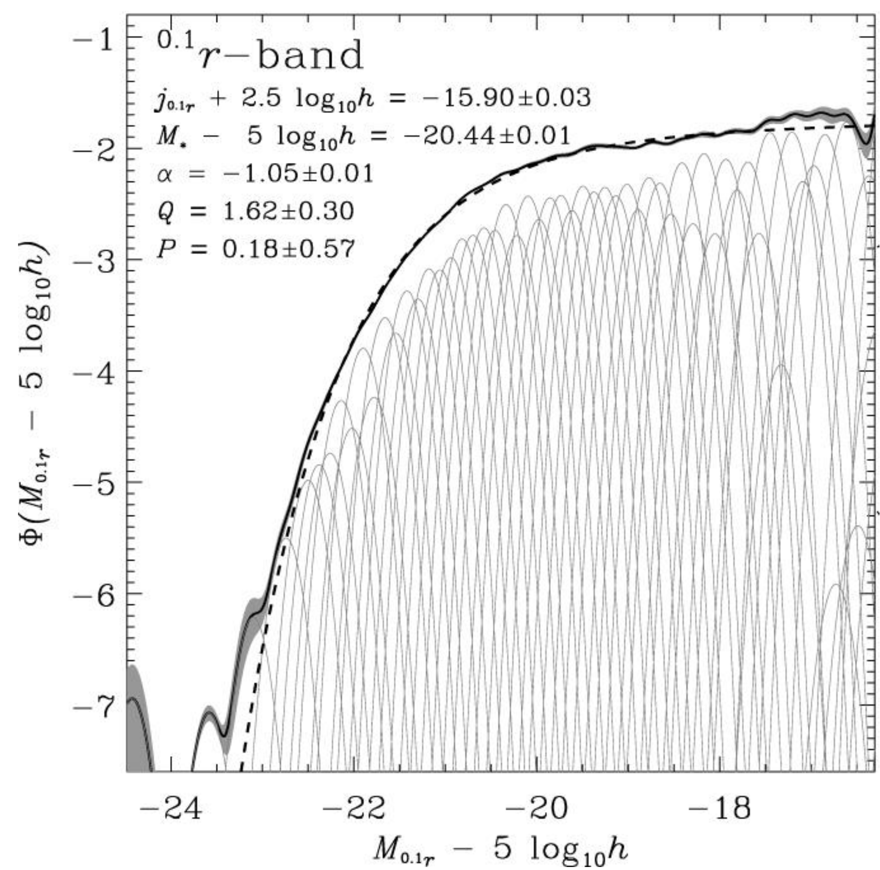
Distribution function

LF of galaxies

- The basic statistical properties of galaxies in any galaxy survey
- Schechter function
  - Characteristic luminosity M_*
  - Faint end slope α

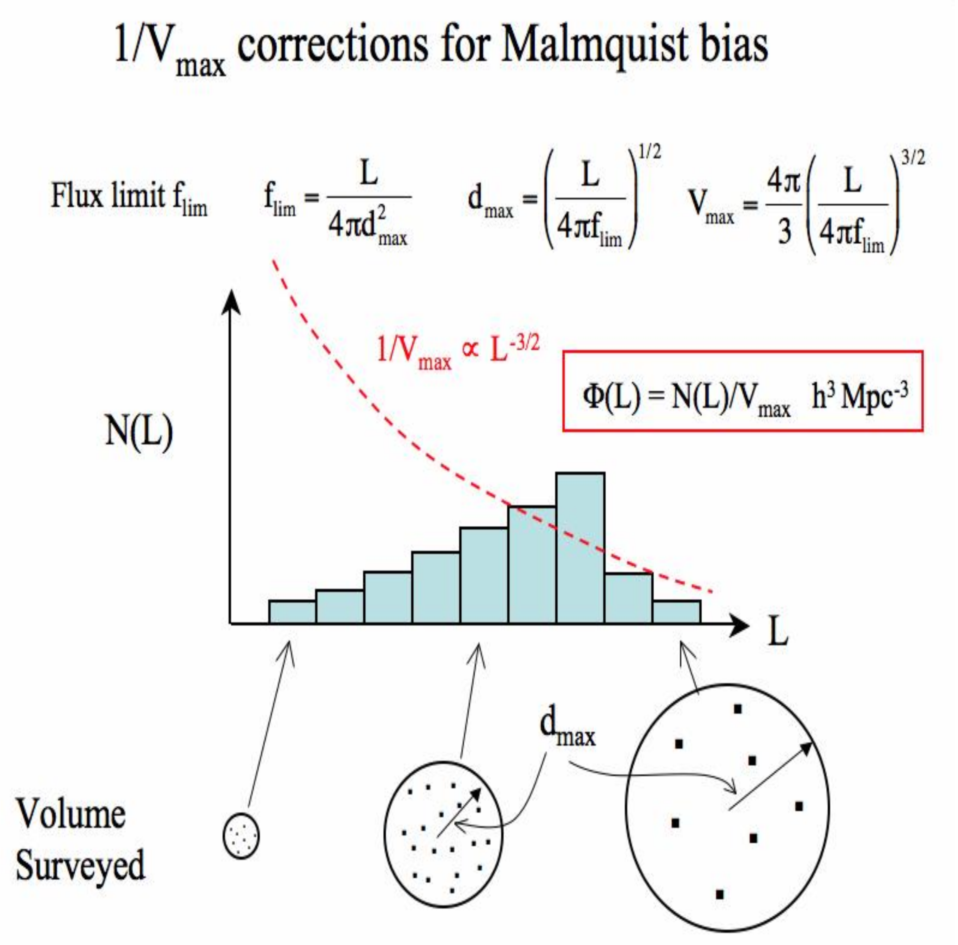
$$\phi(L)dL = \phi^*\left(\frac{L}{L^*}\right)^{\alpha} exp\left(-\frac{L}{L^*}\right)\frac{dL}{L^*}$$

Blanton et al. (2003) (astro-ph/0210215)



Traditional Vmax estimation of LF(Felton 1977)

- Vmax: maximum volume of a galaxy with certain absolute luminosity can be observed in the flux limited sample
  - For flux limit complete sample: <V/Vmax>=0.5
- Advantage: no assumption of the LF shape
- Shortcoming: based on the assumption that galaxy distribution is homogenous



Maximum likelihood estimation

- The probability of a galaxy in the sample

$$p_i = \left(\frac{\Phi(L_i)}{\int_{L_{min}(d_i)}^{\infty} \Phi(L)\, dL} \right)$$

$$\phi(L)dL = \phi^* \left(\frac{L}{L^*}\right)^{\alpha} exp\left(-\frac{L}{L^*}\right)\frac{dL}{L^*}$$

  - $L_{min}(d_i)$, the minimum luminosity above the flux limit.
    - Selection effect
- The likelihood function $\boldsymbol{P} = \prod_i p_i$
- Maximize L as function of M_*, α
  - How to maximize?
    - Analytical: exercise on a Gaussian distribution.
    - numerical calculations in parameter space
  - No direct constraint on ϕ_*

$$\frac{\partial lnP}{\partial \alpha} = 0$$

$$\frac{\partial lnP}{\partial L^*} = 0$$

Step-Wiese Maximum Likelihood method (Efstathiou et al. 1988)

- LF is function of N steps
  - Avoid to use Schechter function as a prior

$$\phi(L) = \phi_k, \quad L \in (L_k - \Delta L/2, L_k + \Delta L/2), \quad k = 1, \ldots, N$$

The likelihood, as in the previous method, then is:

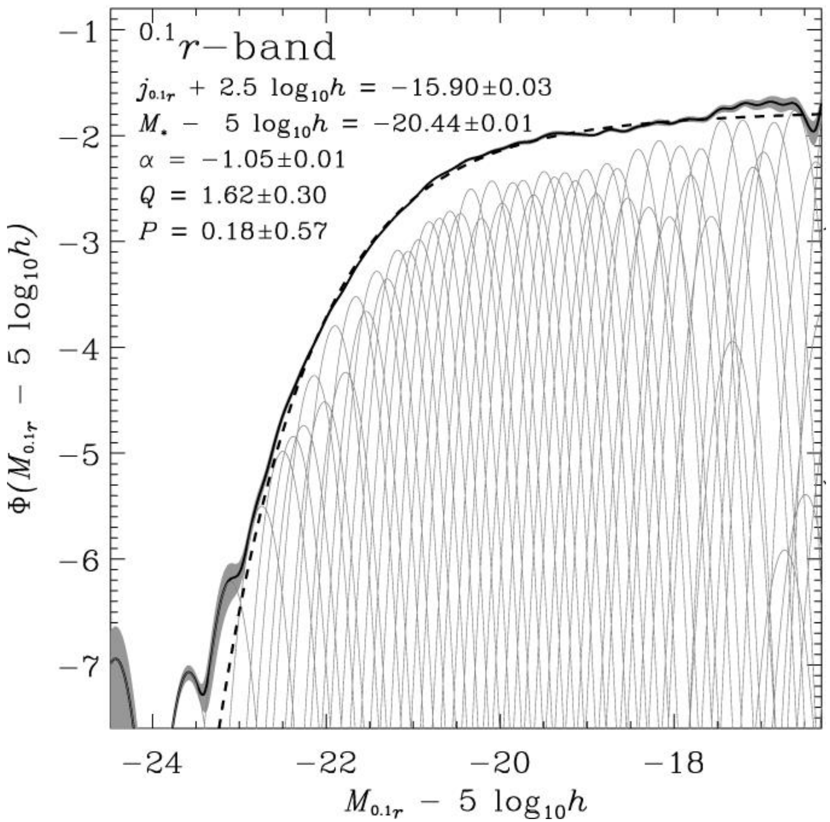
$$\ln L = \sum_{i=1}^{N} W(L_i - L_k) \ln \phi_k - \sum_{i=1}^{N} \ln \{ \sum_{j=1}^{N} \phi_j \Delta L H[L_j - L_{min}(z_i)] \} + C$$

LF estimator of SDSS (Blanton et al. 2003)

$$\Phi(M,z) = \bar{n}10^{0.4(z-z_0)P} \sum_k \Phi_k \frac{1}{\sqrt{2\pi\sigma_M^2}} \times \exp\left\{-\frac{1}{2}\frac{[M - M_k + (z-z_0)Q]^2}{\sigma_M^2}\right\}$$

- Using *n* Gaussian instead of steps
- Considering luminosity evolution (Q)

Blanton et al. (2003) (astro-ph/0210215)



Other methods

Choloniewski method (Choloniewski 1986)

- Consider the selection in the (M,μ) plane together
- Get the normalization

Notes on LF estimation

- Sample completeness is most important
  - Low surface brightness galaxies are always the topic
- Should consider cosmic variance in high redshift survey
- With modern data, conditional LFs are discussed more and more
  - Morphology, color, environment etc.

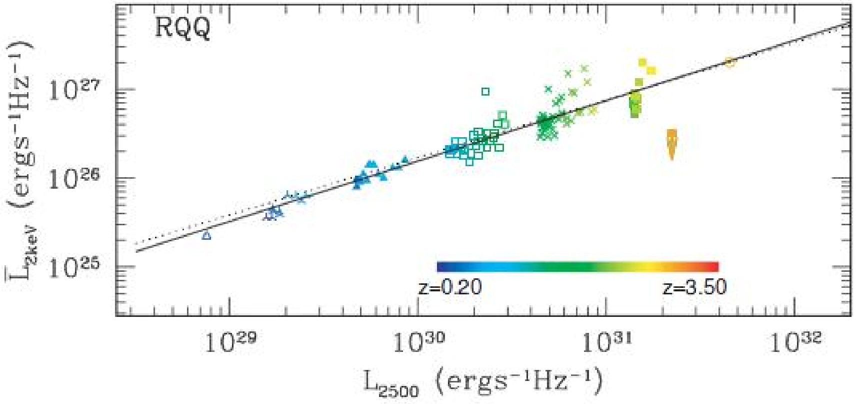
IV: stacking technique

- Only upper limits for very faint source
  - needs deeper exposure
- Upper limit includes information
- Stacking: sources supposed to share similar properties, stacking then is equivalent to increase the exposure time
  - Space → time
  - get average properties
    - Signal may be dominated by few bright sources

Mean VS median

- Mean L_{2KeV} at given L_{2500} in stacking
- Median L_{2KeV} at given L_{2500} in individual linear fitting
  - Fitting in Log L_{2KeV}-Log L_{2500} space
- Scatter of Log L_{2KeV} is ~0.4
  - mean and median difference is a factor of 1.7
- Answer maybe the quasar variability
  - Log-normal

Excellent agreement between stacks and individual detection here is misleading



Solid: data from stacks of QSO.
Dotted: data from individual detection.
Shen et al. 2006

V: Extreme value statistics

- Extreme value populations are easily observed
  - e.g. the brightest group/cluster galaxies, the brightest star of a star cluster
    - Order statistics of the early-type galaxy luminosity function (Dobos & Csabai 2012)
- What can a extreme value tell us ?
  - How unusual are the Shapley Supercluster and the Sloan Great Wall (Sheth & Diaferio 2011)
  - Quantifying the rareness of extreme galaxy clusters (Hotchkiss 2011)
  - An application of extreme value statistics to the most massive galaxy clusters at low and high redshift (Waizmann, Ettori, & Moscardini 2012)
  - Temperature maximum in CMB (coles 1988)

Extreme value statistics

- Three types of extreme value distribution, Depends on the tail shape (Fisher–Tippett–Gnedenko theorem)
  - Weibull(no tail)
    - Lowest temperature
  - Fréchet(flat tail)
    - Money of richest people
  - Gumbel (exponential tail)
    - Height of people
  - Requires sample size N>>1
- Brightest group/cluster galaxy
  - Gumbel distribution?

Extreme value statistics/Order statistics (EVS/OS Dobos & Csabai 2011)

- Cumulative distribution of distribution function $f(x)$

$$F(x) = \int_{-\infty}^{x} f(u)\,\mathrm{d}u.$$

- probability of a number $x < X$

$$P(x < X) = F(X).$$

- N independently drawn numbers $\{x_1, x_2, \ldots, x_N\}$, the probability of $\max\{x_i\} = X_m$

$$P_{\mathrm{m}}(X_{\mathrm{m}}) = P(x_i < X_{\mathrm{m}}) = P^N(x < X_{\mathrm{m}}) = F^N(X_{\mathrm{m}}).$$

- the probability density function of the maximum of a sample of size N

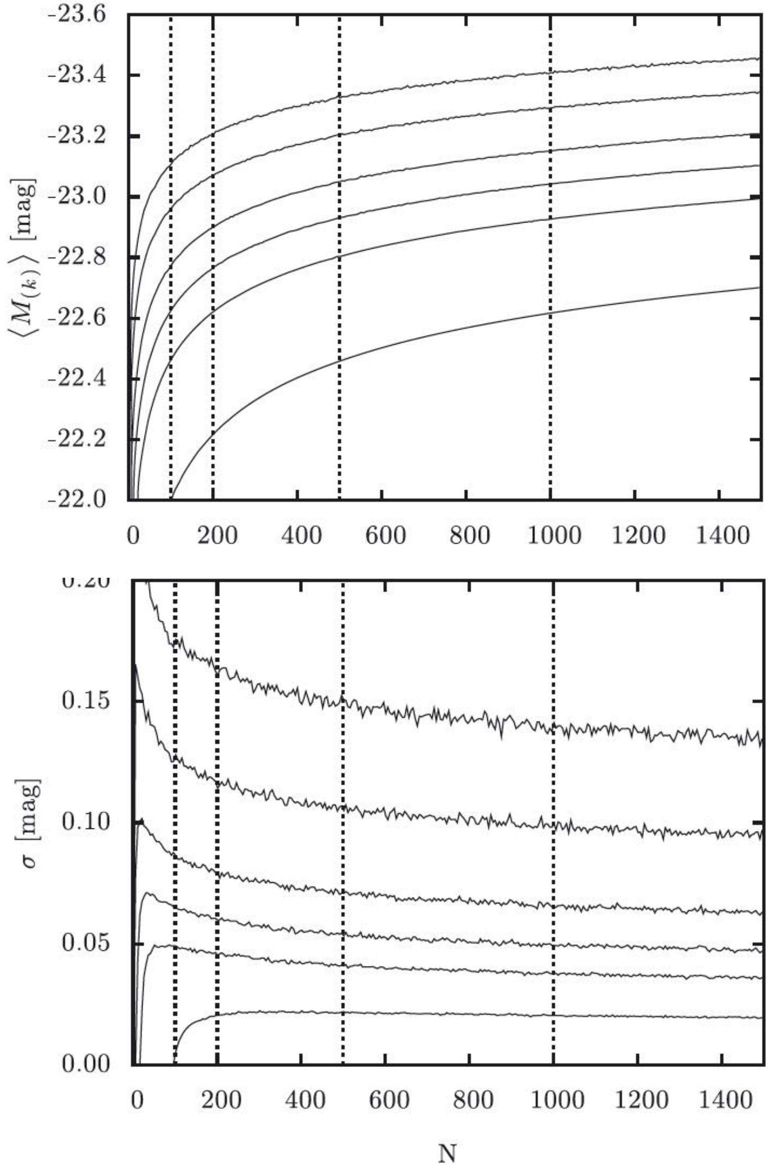
$$p_{\mathrm{m}}(X_{\mathrm{m}}, N) = N F^{N-1}(X_{\mathrm{m}}) f(x).$$

- The probability distribution of the kth largest value

$$p_{(k)}(X_{(k)}, N) = \frac{N!}{(k-1)!(N-k)!}[1 - F(X_{(k)})]^{k-1} F^{N-k}(X_{(k)}) f(X_{(k)}).$$

EVS/OS:
basic conclusions

- The mean extreme values of a lager sample is larger
  - Height of Chinese basket-ball team player is taller than Japanese
  - Brightest galaxies of rich clusters is more luminous than poor groups
- The scatter of the extreme values of a lager sample is smaller
  - BCGs have small scatter
  - The scatter of the higher order members is even smaller



Dobos & Csabai 2011

I. Correlation between Parameters

- Pearson correlation coefficient
  - -1<r<1

$$r = \frac{\sum_i (x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum_i (x_i - \overline{x})^2} \sqrt{\sum_i (y_i - \overline{y})^2}}$$

- Spearman rank: replacing x_i, y_i by the rank R_i, S_i

$$r_s = \frac{\sum_i (R_i - \overline{R})(S_i - \overline{S})}{\sqrt{\sum_i (R_i - \overline{R})^2} \sqrt{\sum_i (S_i - \overline{S})^2}} \qquad (14.6.1)$$

The significance of a nonzero value of r_s is tested by computing

$$t = r_s \sqrt{\frac{N-2}{1-r_s^2}} \qquad (14.6.2)$$

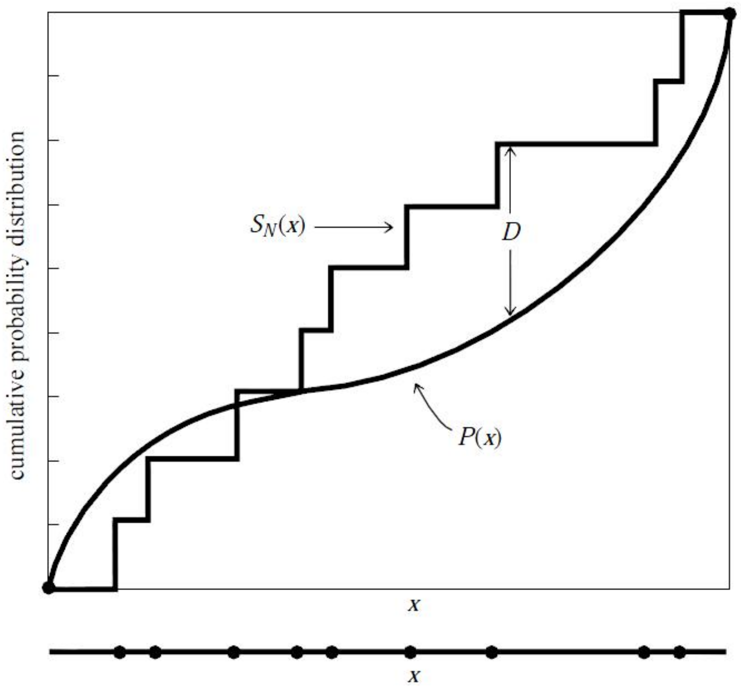
t: Student's distribution with N – 2 degrees of freedom.

K-S test: applicable to unbinned distributions

- *K–S* test defined as the *maximum value* of the absolute difference between two cumulative distribution functions.

$$D = \max_{-\infty < x < \infty} |S_{N_1}(x) - S_{N_2}(x)|$$

$$P(>D) = 2\sum_{i=1}^{\infty}(-1)^{i-1}e^{-2ni^2D^2}$$



- *Can be generalized to two-dimensional distributions*

- *invariant of the parameterization of x*
- *most sensitive around the median*

Is the correlation between A and B real or because A and B are both correlated with C?

Partial correlation

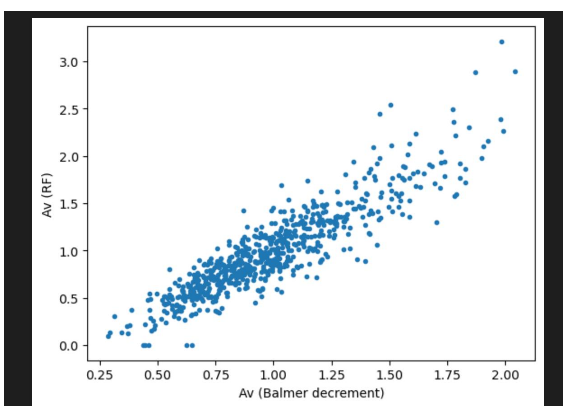
- X correlated with Z, Y correlated with Z, whether X correlated with Y
  - Distance dependent parameters, e.g. L_R VS L_X
- Idea: calculate the correlation between the residuals
  - assumes linear relationship. $$r_{XY \cdot Z} = \frac{r_{XY} - r_{XZ} r_{YZ}}{\sqrt{(1 - r_{XZ}^2)(1 - r_{YZ}^2)}}.$$
- More generalized: multiple regression

Control sample

- We see different b/a values between AGNs and normal spirals. What does it mean? (Shen et al. 2010)
  - b/a is function of stellar mass, size etc.
  - AGNs biased to high stellar mass sample
- We build a control sample of galaxies, which have the same stellar mass, size, concentration, color distributions as AGNs
  - We then compare the b/a of AGNs with control sample

Machine learning: Decesion tree

- Correlation exsit: if we can predict y (output) from x_i(input) by any way
    - $N_{freedom} < N_{data}$
- For lot of X (e.g. stellar mass, size, color, Age, redshift), how they correlate y (e.g. AGN?)
    - the smallest number of dataset x_i that can best predict y
    - which X contribute the most info?



	feature	importance
6	HA_LUM	0.248381
11	star_red	0.215805
0	Met_line	0.119788
9	Age	0.073806
5	EW_HA	0.062337
8	MtoL	0.059757
7	RtoRe	0.054103
10	Meta	0.041470
3	HA_SIGMA	0.041362
1	LOGU	0.038559

II. Linear fitting

$$Y = a\,x + b$$

Famous linear relations in astronomy

- period -luminosity relation of Cepheids
- M_{BH}-σ relation
- Tully-Fisher (L - V_{max}) relation
- Fundamental plane of ellipticals
- L-T, L-σ relation of groups and clusters
- All are statistical scaling relations, none of them are first principle like *F=ma*

Nature of the scaling relations

- Observables: (x_i, y_i) with error ($\Delta_{x,i}$, Δ_{yi})
- First, we should find some correlations, e.g. rank analysis
- To the first order, all the correlations are linear
- $Y = a*X + b + \sigma$
  - σ is the intrinsic scatter, may not be a constant
- Observables maybe biased
  - e.g. some low-luminosity galaxies are not observed at given V_{max}
- Some observables may only be upper limits
  - E.g. we only get the upper-limit of L_x of some cluster

Ordinary Linear regression OLS(y|x)

- y_i with measurement error σ_i

$$\chi^2(a,b) = \sum_{i=1}^{N} \left(\frac{y_i - a - bx_i}{\sigma_i} \right)^2$$

Code: *fit* in numeric recipes

Error on both x and y

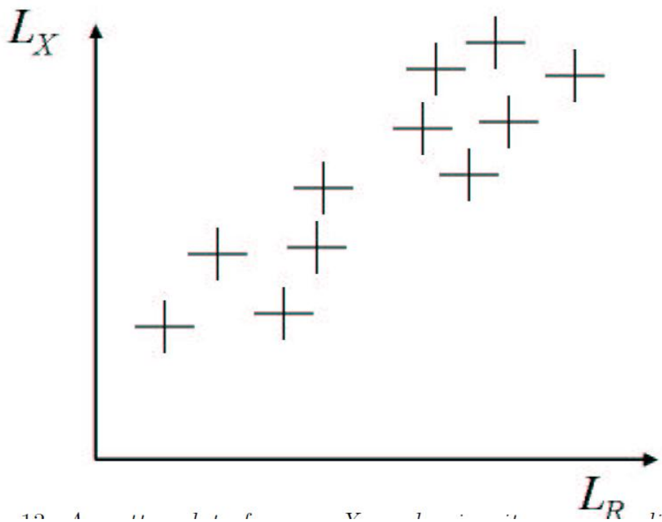
$$\chi^2(a,b) = \sum_{i=1}^{N} \frac{(y_i - a - bx_i)^2}{\sigma_{y\,i}^2 + b^2\sigma_{x\,i}^2}$$

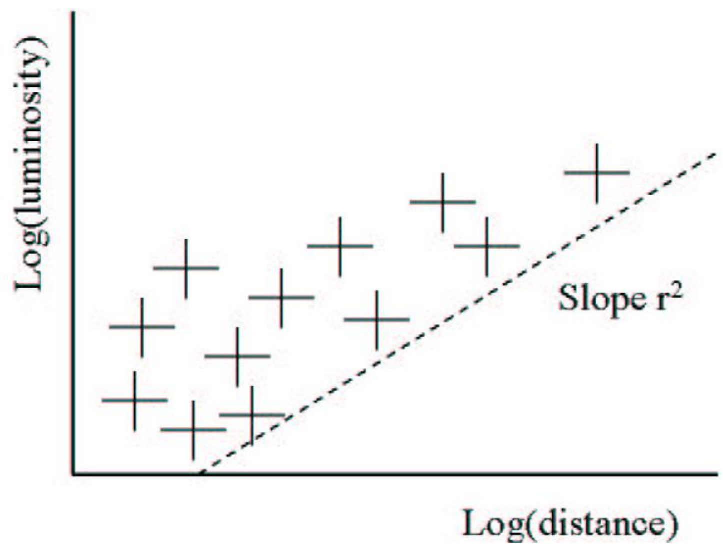
Code: *fitexy* in numeric recipes

b ~ biased to infinity

Eddington(Malmquist) bias

- Distance dependent observable
  - Eddington (1915) Malmquist(1920)
  - In magnitude limit sample, more faint source scattered in than bright source scattered out

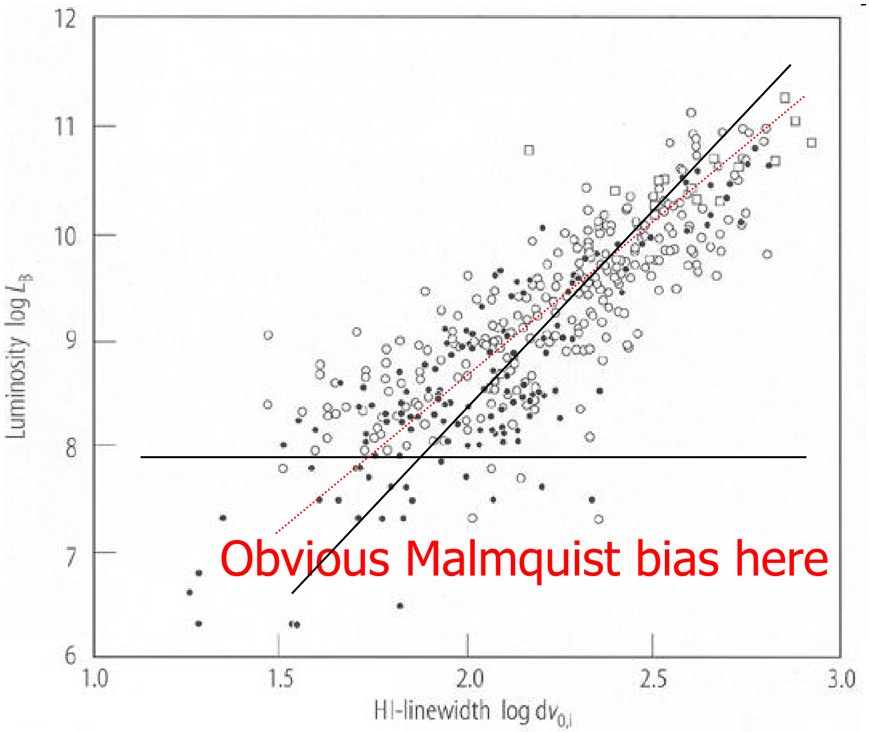




Caveat: choose proper parameterization

- If we fit $M = a \log W + b$, a will be biased to smaller values

- Fit $\log W = a' M + b'$ is better
  - At given M, no obvious in W



Attenuation bias

- "Why Machine Learning Models Systematically Underestimate Extreme Values" arXiv:2412.05806 (Yuan-Sen Ting)

$$y_{true} = \beta x_{true}, \tag{1}$$

$$y_{\text{obs}} = y_{true} + \delta_y, \tag{2}$$

$$\mathbb{E}[\hat{\beta}] = \beta \frac{\sigma^2_{\text{range}}}{\sigma^2_{\text{range}} + \sigma^2_x} = \beta \frac{1}{1 + (\sigma_x/\sigma_{\text{range}})^2}. \tag{10}$$

Six different linear regression

- Reference
  - Linear regression in astronomy I (1990, ApJ,364,104)
    - Different regression method
  - Linear regression in astronomy (1992ApJ...397...55)
    - Truncated, censored data
- IDL code: sixlin
  - Ordinary Least Squares (OLS) Y vs. X (c.f. linfit.pro)
  - Ordinary Least Squares X vs. Y
  - Ordinary Least Squares Bisector
  - Orthogonal Reduced Major Axis ;
  - Reduced Major-Axis
  - Mean ordinary Least Squares

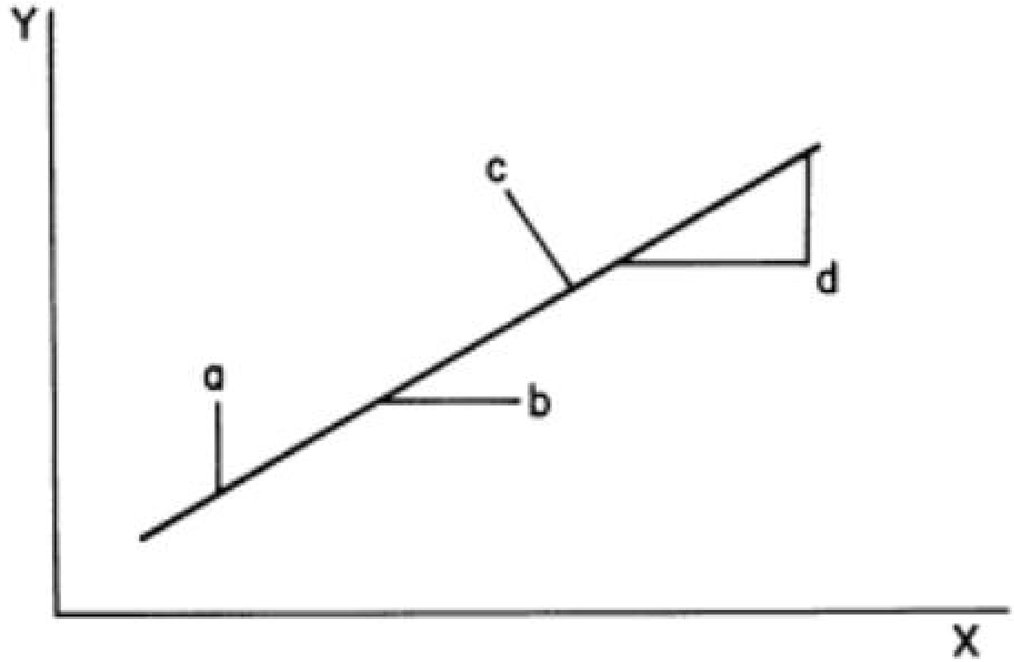


FIG. 1.—Illustration of the different methods for minimizing the distance of the data from a fitted line: (*a*) OLS($Y \mid X$), where the distance is measured vertically; (*b*) OLS($X \mid Y$), where the distance is taken horizontally; (*c*) OR, where the distance is measured vertically to the line; and (*d*) RMA, where the distances are measured both perpendicularly and horizontally. No illustration of the OLS bisector is drawn in this figure.

- The applicability of the procedures is dependent on the nature of the astronomical data under consideration and the scientific purpose of the regression.
- For problems needing symmetrical treatment of the variables, the OLS bisector performs significantly better than orthogonal or reduced major-axis regression.

Error on both x and y and with a constant intrinsic scatter σ

$$\ln L = -\frac{1}{2}\sum_i \ln(\sigma^2 + \sigma_{y,i}^2 + a^2\sigma_{x,i}^2) - \sum_i \frac{[\hat{y}_i - (a\hat{x}_i + b)]^2}{2(\sigma^2 + \sigma_{y,i}^2 + a^2\sigma_{x,i}^2)} + \text{constant.}$$

BCES (Akritas & Bershady, ApJ 470, 706 1996)

- Regression with correlated measurement errors and intrinsic scatter
  - allows for measurement errors on both variables
  - allows the measurement errors for the two variables to be dependent
  - allows the magnitudes of the measurement errors to depend on the measurements
- Intrinsic scatter: constant
- IDL code: BCES.pro (BCES: bivariate, correlate errors and scatter)

Regression for Astronomical Data with Realistic Distributions, Errors and Non-linearity (Tao Jing & Cheng Li)

- arXiv:2411.08747

Table 1. Comparison of Different Regression Methods

Method	$P(\mathbf{x})$	$P(\mathbf{x}_{err}\|\mathbf{x})$	$P(y\|\mathbf{x}, \boldsymbol{\theta})$	Optimization Objective
ML based method (This work)	NF	NF	Any	Likelihood/Posterior
KS-test based method (This work)	NF	NF	Any	p-value of 2D KS test
OLS/WLS	...	...	Linear	Likelihood
ODR/wODR	Uniform	...	Linear	Likelihood
mODR	...	...	Linear	Likelihood
`LINMIX/ROXY`	GMM	...	Linear/Any	Posterior
`Leopy`	Input (heuristic)	...	Any	Likelihood/Posterior
`LtsFit`	...	...	Linear	Likelihood

NOTE—$P(y_{err}|y)$ is modelled by the same method as $P(\mathbf{x}_{err}|\mathbf{x})$.

Normalizing Flows (NF)

Special cases

Robust estimation

- Data with outlier

$$\text{minimize over } \mathbf{a} \qquad \sum_{i=1}^{N} \rho\left(\frac{y_i - y(x_i;\mathbf{a})}{\sigma_i}\right)$$

$$\sum_{i=1}^{N} |y_i - a - bx_i|$$

See Numeric recipes C15.7

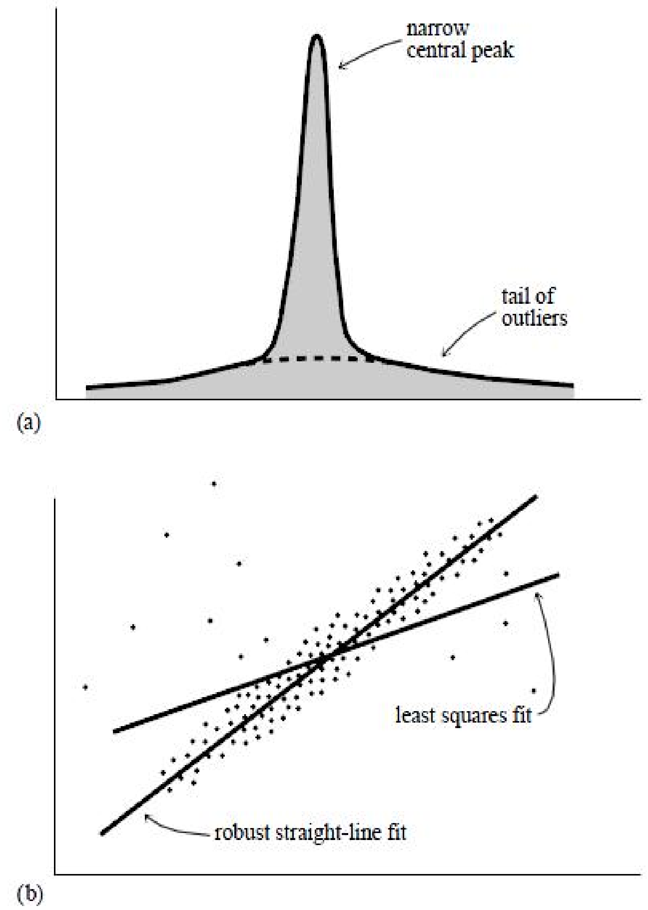
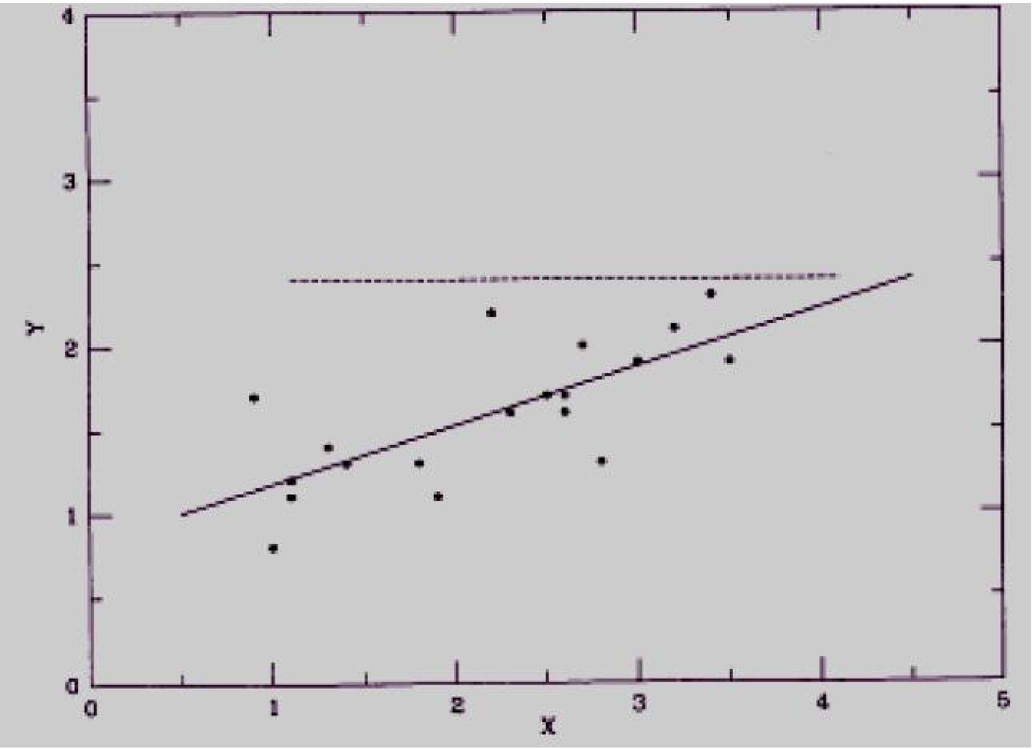


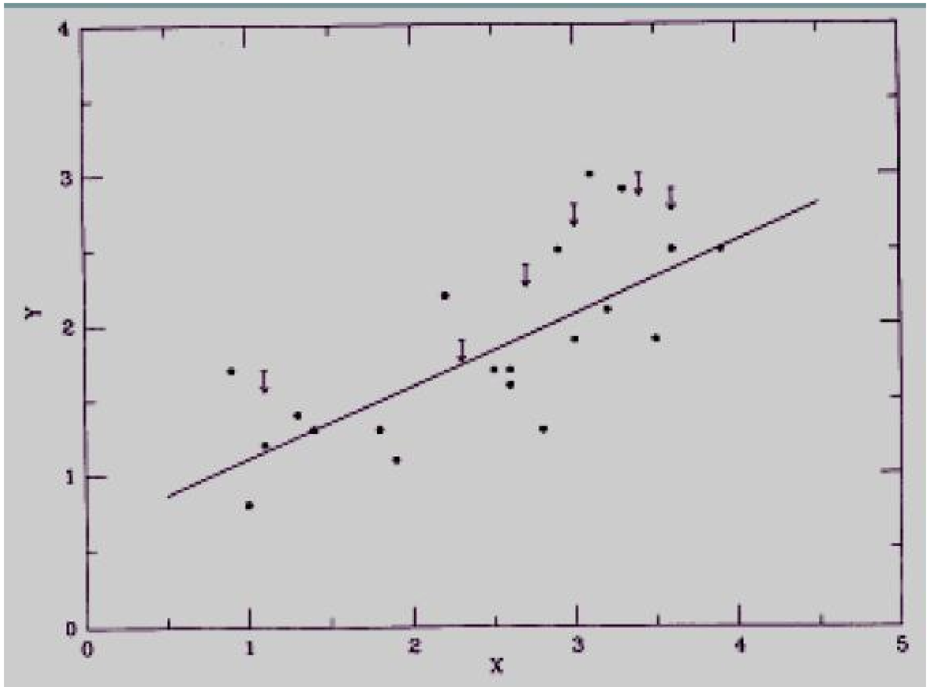
Figure 15.7.1. Examples where robust statistical methods are desirable: (a) A one-dimensional distribution with a tail of outliers; statistical fluctuations in these outliers can prevent accurate determination of the position of the central peak. (b) A distribution in two dimensions fitted to a straight line; non-robust techniques such as least-squares fitting can have undesired sensitivity to outlying points.

Truncation due to flux limits



Malmquist bias in Hubble diagram (Deeming, Vistas Astr 1968, Segal, PNAS 1975)

Censoring due to non-detections



Presented for astronomy by Isobe, Feigelson & Nelson (ApJ 1986)
Implemented in Astronomy Survival Analysis (ASURV) package

最大似然法：假设分布函数f(y|x)

A likelihood function describing a given data set can be defined using the above formulations. Consider a detected point falling in a bin $(z_i, z_i + \Delta z)$. The probability that this occurs is determined by the probability density and is

$$P_D(z_i) \approx f(z_i)\Delta z \ . \tag{10}$$

If an object is right censored at z_i, so that the true location of the point is somewhere between z_i and ∞, the contribution from this point can be written in terms of the survival function

$$P_C(z_j) \approx \int_{z_j}^{\infty} f(t)dt = S(z_j) \ . \tag{11}$$

If there are m detected observations, and n censored observations, the likelihood function is expressed by

$$L = \prod_D^m f(z_i) \cdot \prod_C^n S(z_j)(\Delta z)^m \ ,$$

where $\prod_D^m$ denotes the product over the m detected points, and $\prod_C^n$ denotes the product over the n censored points. Since $(\Delta z)^m$ does not contribute to the maximum, the likelihood can be rescaled to be

$$L = \prod_D^m f(z_i) \prod_C^n S(z_j) \ . \tag{12}$$

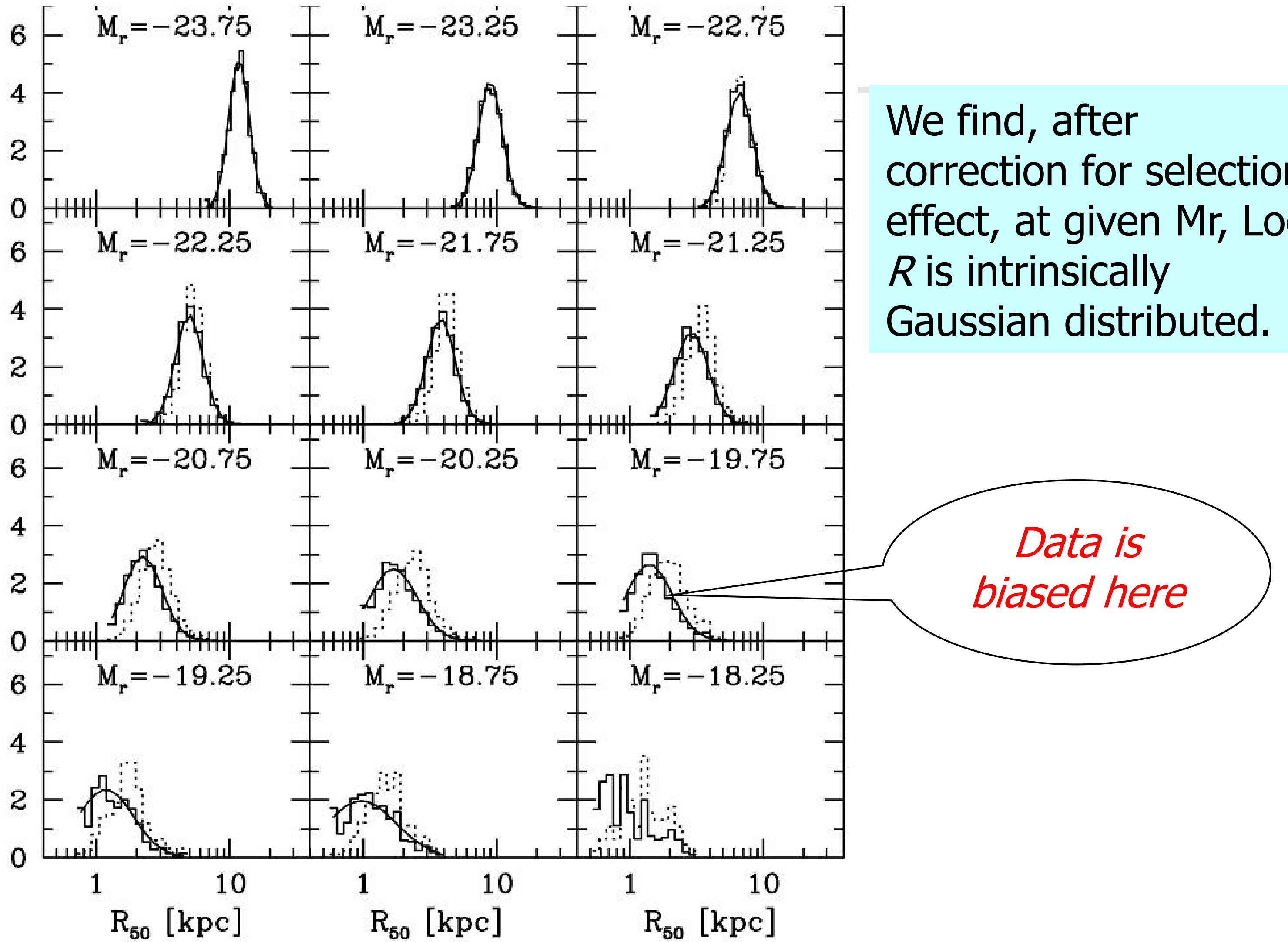
Taking the logarithm, we get the log likelihood function

$$l = \sum_D^m \log f(z_i) + \sum_C^n \log S(z_j) \ . \tag{13}$$

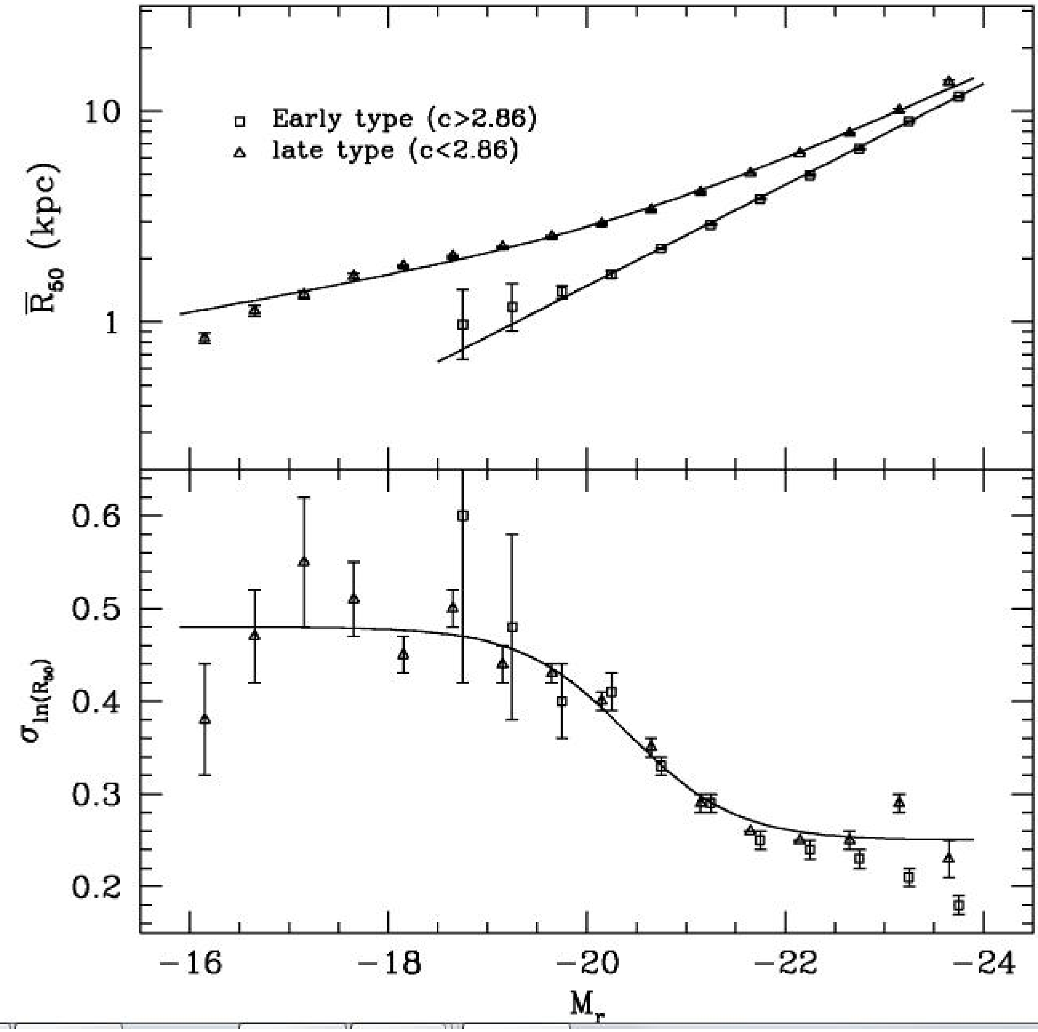
A more straight-forward way

- Especially when amount of data is large in modern surveys
- First, at given bin of x, what is the distribution of y after correction for selection bias?
  - Is y Gaussian distributed? What is the scatter compared with its measurement error?
- Then what is the PDF(y|x) changes as function of x
  - Is this relation linear or non-linear?
- Build the likelihood function and fit the model parameters

L – R relation of galaxies (Shen et al. 2003)



We find, after correction for selection effect, at given Mr, Log *R* is intrinsically Gaussian distributed.



We plot P(R|M) as function of M.

Intrinsic scatter is not a constant

II. Luminosity function of galaxies

Machine learning technic

- 《Statistical Machine Learning for Astronomy》by Yuan-Sen Ting arXiv:2506.12230

Final thoughts

- Use proper model
  - Depend on your question.
  - Question is the first step of your science
- Use proper way to do the statistics
  - Need to know the principle, may need not know the detail.
- Use proper evidence
  - Model explains everything is wrong
  - Depend on your knowledge and experience
- Data mining