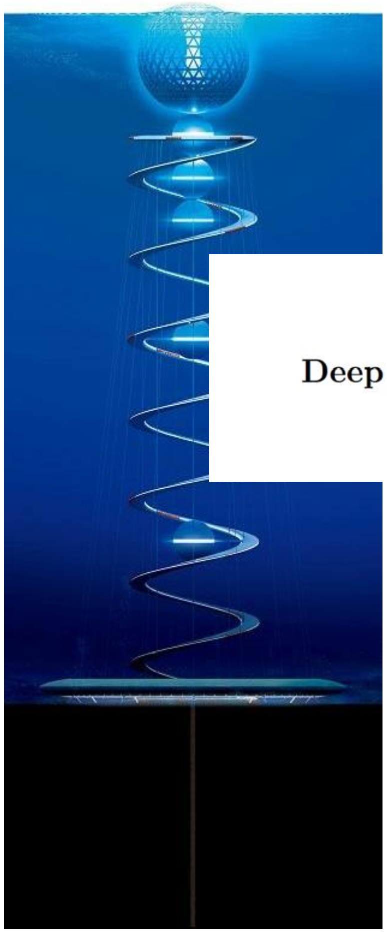

Deep Potential: Recovering the gravitational potential from a snapshot of phase space

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arXiv 2205.02244

Distribution function

$f(\mathbf{x}, \mathbf{v}, t)$ is called the **distribution function** or **phase-space density** of the system. Throughout this book we abbreviate "distribution function" to "DF". Clearly $f \geq 0$ everywhere in phase space.

- 相空间，x1,x2,x3,v1,v2,v3
- 全同粒子体系
  - 理想气体
  - 星系，星团
- 概率函数
  - 标量，可以写成其它变量的函数

$$\nu \equiv \int f d^3\mathbf{v} \quad ; \quad \overline{v}_i \equiv \frac{1}{\nu} \int f v_i d^3\mathbf{v},$$

无碰撞玻尔兹曼方程（CBE）（Collisionaless Boltzmann Equation）

- 连续性方程
  - 不可压缩液体 （概率守恒）

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha} = 0, \tag{4-13a}$$

i.e.,

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right) = 0, \tag{4-13b}$$

or, in vector notation,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \tag{4-13c}$$

恒星系统：恒星的寿命，恒星之间的相关性

The meaning of the collisionless Boltzmann equation can be clarified by extending to six dimensions the concept of the convective or Lagrangian derivative [see eq. (1E-7)]. We define

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha}. \tag{4-14}$$

(df/dt) represents the rate of change of the density of phase points as seen by an observer who moves through phase space with a star at velocity $\dot{\mathbf{w}}$. The collisionless Boltzmann equation (4-13a) is then simply

$$\frac{df}{dt} = 0. \tag{4-13d}$$

稳态系统：分布函数不随时间演化

$$\frac{\partial f}{\partial t} = 0.$$

金斯方程

$$\int \frac{\partial f}{\partial t} d^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i} d^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} d^3\mathbf{v} = 0,$$

$$\nu \frac{\partial \overline{v}_j}{\partial t} - \overline{v}_j \frac{\partial (\nu \overline{v}_i)}{\partial x_i} + \frac{\partial (\nu \overline{v_i v_j})}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j}, \qquad (4\text{-}25)$$

and then noting that the mean value of $v_i v_j$ may be broken into a part $\overline{v}_i \overline{v}_j$ that is due to streaming motion and a part

$$\sigma_{ij}^2 \equiv \overline{(v_i - \overline{v}_i)(v_j - \overline{v}_j)} = \overline{v_i v_j} - \overline{v}_i \overline{v}_j \qquad (4\text{-}26)$$

that arises because the stars near any given point $\mathbf{x}$ do not all have the same velocity. Then using equation (4-26) in equation (4-25), we obtain the analog of Euler's equation (1E-8) of fluid flow;

$$\nu \frac{\partial \overline{v}_j}{\partial t} + \nu \overline{v}_i \frac{\partial \overline{v}_j}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial (\nu \sigma_{ij}^2)}{\partial x_i}. \qquad (4\text{-}27)$$

加速度 旋转 势场 压力

柱坐标下的金斯方程

$$\frac{\nu\overline{v_R^2} - \nu\overline{v_\phi^2}}{R} + \frac{\partial\left(\nu\overline{v_R^2}\right)}{\partial R} + \frac{\partial\left(\nu\overline{v_R v_z}\right)}{\partial z} = -\nu\frac{\partial\Phi}{\partial R}$$

$$\frac{\nu\overline{v_R v_z}}{R} + \frac{\partial\left(\nu\overline{v_z^2}\right)}{\partial z} + \frac{\partial\left(\nu\overline{v_R v_z}\right)}{\partial R} = -\nu\frac{\partial\Phi}{\partial z}$$

We shall see below that equation (4-27) is valuable for its ability to relate observationally accessible quantities, like the streaming velocity, velocity dispersion, and so forth. But its fundamental defect must be recognized: we have no analog of the equation of state of a fluid system to relate the six independent components of the tensor $\boldsymbol{\sigma}^2$ to the density ν.

能不能直接从观测来推算分布函数?

参数化 （运动积分）， 一般情况?

Differentiable approximation of DF

- 观测到的恒星的位置和速度是分布函数的采样~ （注意选择效应）
  - 如果假设分布函数的形式，可以根据观测量用最大似然法把分布函数的参数导出
- 深度学习: 用一堆参数φ的p $_{\varphi}$ (z)来近似DF，这个DF和标准高斯之间存在一个映射 (normalization flow)
  - 训练：本质上还是最大似然，但是这时候重点不是参数的数值，重点是p $_{\varphi}$ (z)可以近似DF, 而且p $_{\varphi}$ (z)是可微的

$$p(\vec{z}) = \mathcal{N}(\vec{z}' \mid 0, \mathbb{1}) \left| \frac{\partial \vec{z}'}{\partial \vec{z}} \right| . \tag{4}$$

The key is to find a parameterized class of highly flexible, nonlinear, bijective transformations, for which the Jacobian can be efficiently calculated. In this paper, we call the parameters governing this family of coordinate transformations φ, and we refer to the resulting probability density function in $\vec{z}$ as $p_\varphi(\vec{z})$. Given a set of points $\{\vec{z}\}$ that are drawn from an unknown distribution $p(\vec{z})$, we can then search for the parameters φ that maximize the likelihood of the points. This then yields a smooth approximation of the distribution from which the points $\{\vec{z}\}$ were drawn. The

$$\frac{\partial f}{\partial t} = \sum_i \left(\frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} - v_i \frac{\partial f}{\partial x_i} \right) = 0 \,. \tag{2}$$

Once we can describe the distribution function of a stationary system, in all physically plausible cases, the gravitational potential can be uniquely determined (up to an additive constant) by solving the Collisionless Boltzmann Equation (see Appendix A, as well as An et al. 2021). Realistic physical systems will not be completely stationary, and as such, there may not exist any potential which would render the system stationary. In general, therefore, Deep Potential recovers the potential which *minimizes* the amount of non-stationarity in the system (using a measure that will be discussed below). Fig. 1 gives a graphical overview of Deep Potential.

Note that we do not assume that the gravitational potential is sourced by the observed stellar population alone. Accordingly, we do not impose the condition

$$\nabla^2 \Phi = 4\pi G \int f\left(\vec{x}, \vec{v}\right) \mathrm{d}^3\vec{v} \,. \tag{3}$$

2.2. *Modeling the gravitational potential*

After learning the distribution function, we find the gravitational potential $\Phi(\vec{x})$ that renders the distribution function stationary. The distribution is stationary when Eq. (2) is satisfied everywhere in phase space. We parameterize the gravitational potential as a feed-forward neural network, which takes a 3-vector, $\vec{x}$, and returns a scalar, Φ. We denote the trainable parameters of this network (i.e., the weights and biases) by θ, and the resulting approximation function as $\Phi_\theta(\vec{x})$. For any given value of θ, we can calculate the non-stationarity of our approximation of the distribution function at any arbitrary point $(\vec{x}, \vec{v})$ in phase space:

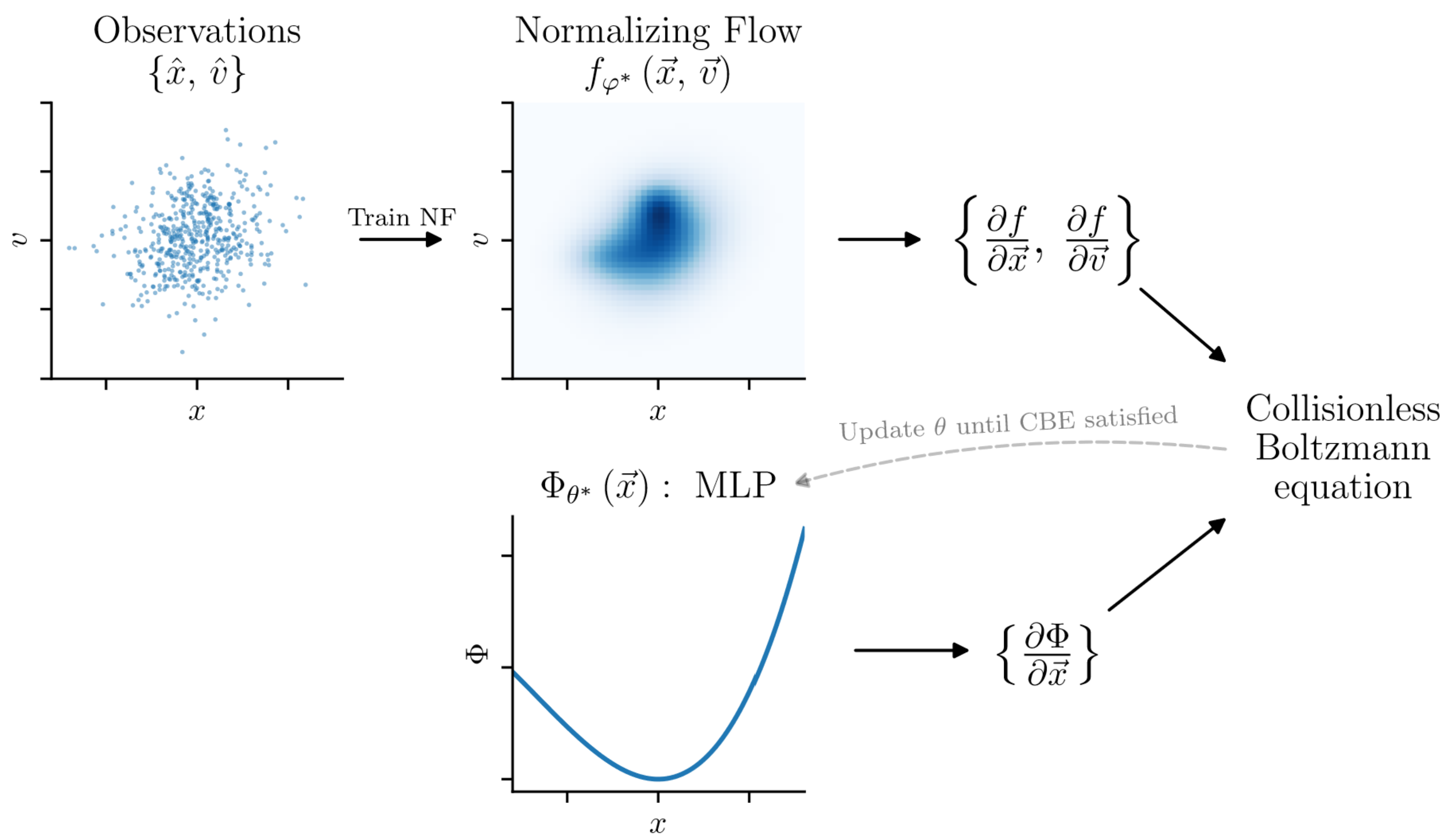
$$\frac{\partial f_{\varphi^*}}{\partial t} = \sum_i \left(\frac{\partial \Phi_\theta}{\partial x_i} \frac{\partial f_{\varphi^*}}{\partial v_i} - v_i \frac{\partial f_{\varphi^*}}{\partial x_i} \right) . \tag{7}$$

This is essentially a variational method, in which our Ansatz for the gravitational potential is a neural network Φ_θ, and in which we vary the parameters θ to minimize non-stationarity of the distribution function.

We require a measure of the non-stationarity of the distribution function throughout all of phase space, which we can then use to find the optimal gravitational potential. We also require that the matter density be non-negative

By Poisson's equation, which links the potential to the density, this implies that $\nabla^2\Phi \geq 0$.

Deep Potential - Method



总结一下

- 恒星系统的引力势可以恒星的分布函数导出（CBE）
  - 没有动力学系统是完全平衡态的
  - 恒星的运动不能简单的理解为引力势中的test particle
  - 恒星的空间分布决定了引力势，引力势决定恒星的运动方程，运动会改变空间分布
- 深度学习方法可以很好的近似复杂方程
  - 其可微分的特性特别适用于解CBE